PERGAMON

International Journal of Multiphase Flow 28 (2002) 127–136

International Journal of Multiphase Flow

www.elsevier.com/locate/ijmulflow

# The onset of particle segregation in plane Couette flows of concentrated suspensions

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Received 23 June 1999; received in revised form 15 August 2001

**Abstract**

We analyze particle segregation in suspensions of rigid particles immersed in Newtonian liquids flowing in plane Couette devices by considering the stability of the system to two-dimensional perturbations in the plane of shear. We obtain the dispersion relation analytically for small wave numbers, finding that when the velocity of the walls is kept constant, the system is always stable and the particle concentration remains uniform. On the other hand, when we keep constant the shear applied at one wall, we predict that a disorder–order transition will occur, provided that $\sigma > 1$, where $\sigma$ is the non-dimensional parameter, $\sigma = (\mu_0'\hat{E}_0)/(\mu_0\hat{D}_0)$. Here $\mu_0$ and $\mu_0'$ are the effective viscosities of the suspension, $\mu$, and its derivative, $d\mu/d\phi$, evaluated at the mean particle volume fraction $\phi_0$, while $\hat{D}_0$ and $\hat{E}_0$ are two transport coefficients appearing in the constitutive relation for the material flux. Substituting experimental correlations for $\hat{D}_0$ and $\hat{E}_0$ into this condition, we conclude that for no values of $\phi_0$ will particles cluster along the plane of shear. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords:* Suspensions; Particle Segregation

## 1. Introduction

The motivation of this work is to analyze the possibility that sheared concentrated suspensions of rigid particles immersed in Newtonian fluids could segregate and form clusters, even when the suspended particles are large enough that the effects of Brownian motion and interparticle potentials can be neglected. For concentrated, charge- and sterically stabilized suspensions, there

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PII: S0301-9322(01)00057-X

appears to be a strong agreement that an order–disorder transition occurs in the suspension microstructure, resulting, in turn, in a shear-thickening macroscopic behavior (see Dratler et al., 1998 and references therein). For suspensions of non-colloidal particles, cluster formation has been observed in Stokesian dynamics simulations by Brady and Bossis (1988), Chang and Powell (1993), Boersma et al. (1995) and others, although no direct observations of this phenomenon had been reported prior to the recent experimental study by Tirumkudulu et al. (1999). There, it was shown that when monodisperse, neutrally buoyant spheres are suspended in a Newtonian liquid and sheared in a partially filled Couette device, they separate into alternating regions of high and low particle concentrations along the longitudinal axis of the cylinder. On the other hand, when the Couette apparatus is completely filled, no particle segregation is observed and the suspension remains uniform.

Phenomenologically, it is well known that suspensions which are initially uniform tend to assume non-uniform concentration distributions when they are subjected to non-uniform shear fields, as shown by Eckstein et al. (1977), Gadala-Maria and Acrivos (1980), Leighton and Acrivos (1987a), Graham et al. (1991), Chow et al. (1994), Koh et al. (1994), Hampton et al. (1997) and others. Many models have been proposed to account for this effect (Leighton and Acrivos, 1987b; Phillips et al., 1992; Nott and Brady, 1994); despite their differences, they all agree in predicting that particles tend to migrate from regions of high concentration to low and from regions of high shear rate to low. Clearly, these two transport mechanisms have, respectively, a stabilizing and destabilizing effect and, whenever the latter prevails, instability, i.e. cluster formation, will occur. Consider, for example, a suspension sheared in a Couette device with constant torque: since the tangential stress is constant within the cross-section, any local increase in the particle volume fraction corresponds to a decrease in the local shear rate, resulting from an increase in the suspension effective viscosity. Therefore, on one hand particles will tend to migrate out of the region of higher volume fraction, while on the other hand they will tend to move in, due to the lower local shear rate: whenever the latter mechanism prevails, instability will occur. This physical model has been first proposed by Nott and Brady (1994) and by Goddard (1997, 1998) for the case of an infinitely extended suspension. Here, we will extend this study to a case where the suspension is confined between two parallel plates.

No complete ab initio analytical theory of shear-induced migration has been developed to date, with the exception of the studies by Wang et al. (1996, 1998), who evaluated the self- and gradient diffusivity of spheres in a dilute suspension, subjected to a constant shear rate. Phenomenologically, Leighton and Acrivos (1987b) and Phillips et al. (1992) have proposed the following constitutive relation for the material flux $\mathbf{J}$ in a suspension of spheres with radius a,

$$\mathbf{J} = -a^2\widehat{D}(\phi)\gamma\nabla\phi - a^2\widehat{E}(\phi)\nabla\gamma, \tag{1}$$

where $\phi$ is the particle volume fraction and $\gamma$ the shear rate, while $\widehat{D}(\phi)$ and $\widehat{E}(\phi)$ are scalar dimensionless concentration-dependent transport coefficients that can be estimated from the experimental results of Leighton and Acrivos (1987a), and Tetlow et al. (1998). The constitutive equation (1) has been used to correctly predict some phenomena occurring in concentrated suspensions such as viscous resuspension (Leighton and Acrivos, 1986; Acrivos et al., 1993; Tirumkudulu et al., 1999) and aberrant shear thinning effects (Acrivos et al., 1994). Its physical meaning is that suspended particles tend to move from regions with high to low collision rates. In this respect, the phenomenon of shear-induced particle diffusion is similar to that of molecular diffusion, the difference being that the latter is due to the interactions of the Brownian particles with the fluid molecules, while the former is the result of particle–particle interactions. In the following, we will denote these interactions by "collisions", although "encounters" is a more fitting name, to stress the analogy with the kinetic theory of gases. Just as in gas diffusion, the mass flux $\mathbf{J}$ is the product of the local concentration by the mass-average particle velocity, with the latter, in turn, depending on the variation of the collision frequency $Z$ over a distance of the order of the mean free path, i.e.

$$\mathbf{J} = -a^2\beta(\phi)\nabla Z,$$

where $\beta(\phi)$ is the ratio between the mean free path and the particle radius. Now, the collision frequency $Z$ is proportional to the local shear rate via a concentration-dependent coefficient $\alpha(\phi)$, i.e. $Z = \alpha(\phi)\gamma$. In particular, in the dilute limit, since no net displacement will result from a two-particle encounter, an effective encounter must involve three or more particles, and therefore, $\alpha(\phi)$ must be proportional to $\phi^2$ as $\phi \to 0$. So, from the above considerations we can conclude that the constitutive relation for the material flux $\mathbf{J}$ reads:

$$\mathbf{J} = -a^2\beta(\phi)\nabla[\alpha(\phi)\gamma], \tag{2}$$

from which we easily obtain Eq. (1).

Now we have to specify what we mean by shear rate $\gamma$. Since $\gamma$ must be invariant to any rotation of the reference frame, it must depend on the three invariants of the shear rate tensor $\mathbf{S}$,

$$\mathbf{S} = \tfrac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\dagger}],$$

where $\mathbf{v}$ is the mass-average suspension velocity, and the dagger indicates the transpose of a tensor. In our case, the first invariants of $\mathbf{S}$ are identically zero, since the fluid is incompressible. In addition, as we will consider two-dimensional flow fields, the third invariant is equal to zero as well, so that we can assume

$$\gamma = \sqrt{2\mathbf{S} : \mathbf{S}}. \tag{3}$$

## 2. Equations of motion and preliminary considerations

Consider a suspension of rigid, neutrally buoyant particles of size $a$, immersed in a Newtonian fluid flowing in the longitudinal $x$-direction through a plane Couette device of thickness $L$. Two cases are considered here, in which either the velocity or the shear stress at the upper plate $y = L$ is fixed, while the bottom plate does not move. The particle size is large enough to prevent molecular diffusion, but it is also small enough compared to the gap thickness, so that the suspension can be treated as a continuum. Since the solvent fluid is Newtonian, we know that the suspension must also behave as a Newtonian fluid, with effective viscosity $\mu(\phi)$ strongly dependent on the local particle volume fraction $\phi$. The equations describing the motion of the suspension are

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \nabla\cdot\mathbf{T}, \tag{4}$$

$$\nabla\cdot\mathbf{v} = \mathbf{0}, \tag{5}$$

where $\rho$ is the suspension density (which here is constant), $\mathbf{v}$ the mass-average suspension velocity, and $\mathbf{T}$ is the stress tensor, which satisfies the following constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{S}, \tag{6}$$

with $p$ denoting the pressure. The equations of motion (4)–(6) are coupled to the following mass conservation equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = -\nabla \cdot \mathbf{J}, \tag{7}$$

where the mass flux $\mathbf{J}$ is given by the constitutive equation (1).

The governing equations (4)–(7) are supplemented with the no-flux boundary conditions,

$$\widehat{\mathbf{1}}_y \cdot \mathbf{J} = 0 \quad \text{at } y = 0 \text{ and } y = L, \tag{8}$$

where $\widehat{\mathbf{1}}_y$ is the unit vector in the transversal $y$-direction and the no-slip boundary condition at the bottom plate,

$$\mathbf{v} = \mathbf{0} \quad \text{at } y = 0. \tag{9}$$

In addition, when a constant velocity $U_0$ is imposed on the upper plate, the boundary condition is

$$\mathbf{v} = U_0 \widehat{\mathbf{1}}_x \quad \text{at } y = L, \tag{10}$$

where $\widehat{\mathbf{1}}_x$ is the unit vector in the longitudinal $x$-direction. On the other hand, when the upper plate is subjected to a constant force per unit area, $F_0$, the boundary condition reads:

$$\widehat{\mathbf{1}}_y \cdot \mathbf{v} = 0 \quad \text{and} \quad \widehat{\mathbf{1}}_y \cdot \mathbf{T} \cdot \widehat{\mathbf{1}}_x = F_0 \qquad \text{at } y = L. \tag{11}$$

Finally, we assume that initially the system is in its steady state, with constant O(1) concentration $\phi_0$ and linear velocity profile, i.e.,

$$\phi = \phi_0, \quad \mathbf{v} = \gamma_0 y \widehat{\mathbf{1}}_x, \quad p = 0 \quad \text{at } t = 0, \tag{12}$$

where $\gamma_0 = U_L/L$ is the unperturbed shear rate. Here $U_L$ denotes the unperturbed velocity of the upper plate, which equals either $U_0$ or $F_0 L/\mu(\phi_0)$, according to whether the fixed velocity or the fixed stress boundary conditions are considered, respectively.

Now we assume that the system is homogeneous in the $z$-direction, so that the problem is two-dimensional in the $xy$-plane. Therefore, considering infinitesimally small disturbances to the base state,

$$\phi = \phi_0 + \widehat{\phi}; \quad \mathbf{v} = \left(\frac{U_L}{L} y + u\right) \widehat{\mathbf{1}}_x + v \widehat{\mathbf{1}}_y, \tag{13}$$

with $\widehat{\phi} \ll \phi_0$, $u, v \ll U_L$, Eq. (3) gives at leading order:

$$\gamma = \frac{U_L}{L} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{14}$$

Now, defining the following non-dimensional variables,

$$\widehat{x} = \frac{x}{L}, \quad \widehat{y} = \frac{y}{L}, \quad \widehat{t} = \frac{tU_L}{L}, \quad \widehat{u} = \frac{u}{U_L}, \quad \widehat{v} = \frac{v}{U_L}, \quad \widehat{p} = \frac{p}{\rho U_L^2}, \tag{15}$$

and linearizing the governing equations, we obtain

$$N_{\mathrm{Re}}(\widehat{u}_t + \widehat{y}\widehat{u}_x + \widehat{v} + \widehat{p}_x) = \widehat{u}_{xx} + \widehat{u}_{yy} + \widehat{\mu}'_0\widehat{\phi}_y, \tag{16}$$

$$N_{\mathrm{Re}}(\hat{v}_t + \hat{y}\hat{v}_x + \hat{p}_y) = \widehat{v}_{xx} + \widehat{v}_{yy} + \widehat{\mu}'_0\widehat{\phi}_x, \tag{17}$$

$$\widehat{u}_x + \widehat{v}_y = 0, \tag{18}$$

$$\widehat{\phi}_t + \widehat{y}\widehat{\phi}_x = \lambda^2\widehat{D}_0(\widehat{\phi}_{xx} + \widehat{\phi}_{yy}) + \lambda^2\widehat{E}_0\left(\widehat{u}_{xxy} + \widehat{u}_{yyy} + \widehat{v}_{xxx} + \widehat{v}_{xyy}\right), \tag{19}$$

where $\lambda = a/L$, $\widehat{\mu} = \mu(\phi)/\mu_0$ and $\widehat{\mu}'_0 = [\mathrm{d}\widehat{\mu}/\mathrm{d}\phi]_{\phi=\phi_0}$, while $N_{\mathrm{Re}} = \rho U_L L/\mu_0$ denotes the Reynolds number based on the thickness of the channel, which can be an O(1) quantity. The subscript "0", as in $\mu_0$, $\widehat{D}_0$ and $\widehat{E}_0$, indicates that the quantity is evaluated assuming that $\phi = \phi_0$, e.g. $\mu_0 = \mu(\phi_0)$, while the subscripts "$t$", "$x$" and "$y$" denote partial derivatives with respect to $\widehat{t}$, $\widehat{x}$ and $\widehat{y}$, respectively, e.g. $\widehat{u}_x = \partial\widehat{u}/\partial\widehat{x}$.

Now, if we express the small disturbances to the base state as a superposition of normal modes,

$$\left[\widehat{u}, \widehat{v}, \widehat{p}, \widehat{\mu}'_0\widehat{\phi}\right]\left(\widehat{x}, \widehat{y}, \widehat{t}\right) = [U, V, P, \Phi](\widehat{y}, k)\exp\left[\mathrm{i}k\left(\widehat{x} - \widehat{c}\widehat{t}\right)\right], \tag{20}$$

substituting (20) into (16)–(19), deriving $U$ and $P$ in terms of $V$ and $\Phi$, and rearranging the resulting equations, we obtain

$$\mathrm{i}kN_{\mathrm{Re}}(\widehat{y} - \widehat{c})(V'' - k^2V) = V'''' - 2k^2V'' + k^4V - \mathrm{i}k(\Phi'' + k^2\Phi), \tag{21}$$

$$\frac{1}{\lambda^2\widehat{D}_0}k^2(\widehat{y} - \widehat{c})\Phi = \sigma\left(V'''' - k^4V\right) - \mathrm{i}k(\Phi'' - k^2\Phi), \tag{22}$$

where prime indicates total derivative with respect to $\widehat{y}$, e.g. $V' = \mathrm{d}V/\mathrm{d}\widehat{y}$, while we have defined the parameter

$$\sigma = \frac{\widehat{\mu}'_0\widehat{E}_0}{\widehat{D}_0}. \tag{23}$$

Note that the separation of variables solution (20) assumes periodic disturbances along the flow direction $x$, as is usual in linear stability studies (see Drazin and Reid, 1981), so that the spatial means of perturbation quantities are identically zero. When an exact solution for the eigenvalue problem is not available, as is the case in the present study, we can proceed analytically to obtain asymptotic expressions for the growth rate by analyzing (21) and (22) for long (or short) waves, as is done in the following section.

## 3. Long wavelength limit

Eqs. (21) and (22) constitute a sixth-order system of two coupled differential equations of the Orr-Sommerfeld type. As expected, the problem looks similar to the study of the stability of pressure-driven resuspension flows in channels (Zhang et al., 1992a), and of buoyancy-driven flows in inclined settlers (Zhang et al. (1992b)). In this chapter, we will solve this problem in the limit of long wavelengths, i.e. when $k \ll 1$, expanding the solutions (20) as

$$[V(\widehat{y},k), \Phi(\widehat{y},k), \widehat{c}(k)] = \sum_{n=0}^{\infty} k^n [V_n(\widehat{y}), \Phi_n(\widehat{y}), \widehat{c}_n]. \tag{24}$$

The two boundary conditions of fixed velocity or fixed stress will be considered separately. Note that in this perturbation scheme the $x$-dependent parts of the eigenfunctions are not expanded in $k$ (see Chandrasekhar, 1961; Drazin and Reid, 1981 and references therein).

### 3.1. Fixed velocity case

In this case, among the six boundary conditions (9)–(11), the first four are

$$V = V' = 0 \quad \text{at } \widehat{y} = 0 \text{ and } \widehat{y} = 1, \tag{25}$$

which can be easily derived from the no-slip boundary conditions $\widehat{u} = \widehat{v} = 0$ at $\widehat{y} = 0$ and $\widehat{y} = 1$, and from the continuity equation, $V' = -\mathrm{i}kU$. The remaining two boundary conditions express the zero mass flux conditions, $\widehat{D}_0 \widehat{\phi}_y + \widehat{E}_0 \widehat{\gamma}_y = 0$ at the walls $\widehat{y} = 0$ and $\widehat{y} = 1$, where $\widehat{\gamma} = 1 + \widehat{u}_y + \widehat{v}_x$. Therefore these conditions give

$$\sigma V''' - \mathrm{i}k\Phi' = 0 \quad \text{at } \widehat{y} = 0 \text{ and } \widehat{y} = 1. \tag{26}$$

Now, substituting expansion (24) into (21), (22), (25) and (26), at O(1) and O($k$), we find

$$V_0 = V_1 = 0, \quad \Phi_0 = \text{constant} = W. \tag{27}$$

At O($k^2$), obtain

$$V_2'''' - \mathrm{i}\Phi_1'' = 0, \tag{28}$$

$$\sigma V_2'''' - \mathrm{i}\Phi_1'' = \frac{1}{\lambda^2 \hat{D}_0}(\widehat{y} - \widehat{c}_0)\Phi_0, \tag{29}$$

with boundary conditions

$$V_2 = V_2' = \sigma V_2''' - \mathrm{i}\Phi_1' = 0 \quad \text{at } \widehat{y} = 0 \text{ and } \widehat{y} = 1. \tag{30}$$

First, integrating Eq. (29) once and comparing the result with the last of the boundary conditions (30) we find the solvability condition $(1 - \widehat{c}_0)^2 = \widehat{c}_0^2$, that is

$$\widehat{c}_0 = \tfrac{1}{2}. \tag{31}$$

This result shows that the perturbations travel along the $x$-axis with the average velocity of the base flow. In addition, integrating Eq. (28) once and substituting (30) we obtain

$$\Phi_1'(0) = \Phi_1'(1), \quad V_2'''(0) = V_2'''(1). \tag{32}$$

Finally, Eqs. (28) and (29) can be integrated, obtaining

$$\Phi_1(\widehat{y}) = -\mathrm{i}\Gamma\left[\frac{1}{6}\xi^3 + \left(\frac{\sigma}{10} - \frac{1}{8}\right)\xi + C\right], \tag{33}$$

$$V_2(\widehat{y}) = \Gamma\left[\frac{1}{120}\xi^5 - \frac{1}{240}\xi^3 + \frac{1}{1920}\xi\right], \tag{34}$$

where $\xi = \widehat{y} - \frac{1}{2}$, while

$$\Gamma = \frac{1}{\sigma - 1}\frac{\widehat{\mu}_0' W}{\lambda^2 \widehat{D}_0}. \tag{35}$$

The constant of integration in Eq. (33) turns out to be irrelevant here; anyway, it can be easily determined through the normalization condition $\int \Phi_1 \mathrm{d}\widehat{y} = 0$, finding that $C = 0$.

Next, at the $\mathrm{O}(k^3)$, we obtain

$$V_3'''' - \mathrm{i}\Phi_2'' = \mathrm{i}N_{\mathrm{Re}}\xi V_2'' + \mathrm{i}\Phi_0, \tag{36}$$

$$\sigma V_3'''' - \mathrm{i}\Phi_2'' = \frac{1}{\lambda^2 \widehat{D}_0}(\xi\Phi_1 - \widehat{c}_1\Phi_0) - \mathrm{i}\Phi_0, \tag{37}$$

to be solved with boundary conditions,

$$V_3 = V_3' = \sigma V_3''' - \mathrm{i}\Phi_2' = 0 \quad \text{at } \widehat{y} = 0 \text{ and } \widehat{y} = 1. \tag{38}$$

Integrating Eq. (37) once and applying the last of the boundary conditions (38), we easily find the solvability condition

$$\widehat{c}_1 = -\mathrm{i}\lambda^2\widehat{D}_0 + \frac{1}{W}\int_0^1 \left(\widehat{y} - \frac{1}{2}\right)\Phi_1 \, \mathrm{d}\widehat{y}. \tag{39}$$

Finally, substituting expression (33) for $\Phi_1$, we obtain

$$\widehat{c}_1 = -\mathrm{i}\left(\lambda^2\hat{D}_0 + \frac{1}{120\lambda^2\widehat{D}_0}\right). \tag{40}$$

Since stability requires that

$$\Im(\widehat{c}) \leqslant 0, \tag{41}$$

we see that the system is always stable.

### 3.2. Fixed stress case

In this case, the boundary condition $\widehat{v} = 0$ at $\widehat{y} = 1$ is replaced by $\widehat{u}_y + \widehat{v}_x = 0$. Substituting expansion (24) into this expression, and using the continuity equation $V' = -\mathrm{i}kU$, we obtain

$$V'' + k^2 V = 0 \quad \text{at } \widehat{y} = 1. \tag{42}$$

The other boundary relations are the no-slip conditions

$$V = V' = 0 \text{ at } \widehat{y} = 0, \quad V = 0 \text{ at } \widehat{y} = 1, \tag{43}$$

and the no-flux condition (26).

Proceeding as in the previous section, at O(1) and $\mathrm{O}(k)$, we find again $V_0 = V_1 = 0$, and $\Phi_0 = \text{constant} = W$, while the solvability condition at $\mathrm{O}(k^2)$ gives again $\widehat{c}_0 = 1/2$. Solving for $V_2$ and $\Phi_1$ and using the fixed stress boundary condition $V_2''(1) = 0$, we obtain

$$\Phi_1(\hat{y}) = -\mathrm{i}\Gamma\left[\frac{1}{6}\xi^3 + \left(\frac{7\sigma}{80} - \frac{1}{8}\right)\xi + C\right], \tag{44}$$

$$V_2(\hat{y}) = \Gamma\left[\frac{1}{120}\xi^5 - \frac{1}{160}\xi^3 - \frac{1}{960}(2\xi^2 - \hat{y}^2)\right], \tag{45}$$

where $C = 0$ when the normalization condition $\int \Phi_1 \mathrm{d}\hat{y} = 0$ is applied. At $\mathrm{O}(k^2)$ the solvability condition gives again Eq. (39). Finally, substituting (44) into (39), we find

$$\hat{c}_1 = -\mathrm{i}\left(\lambda^2 \hat{D}_0 + \frac{1}{120\lambda^2 \hat{D}_0}\frac{1 - \frac{7}{8}\sigma}{1 - \sigma}\right). \tag{46}$$

When $\lambda \ll 1$ we can neglect the $\mathrm{O}(\lambda^2)$ term in Eq. (46), so that the stability condition (41) gives $\sigma < 1$ or $\sigma > \frac{8}{7}$. This means that whenever

$$1 < \sigma < \tfrac{8}{7}, \tag{47}$$

the system is unstable and particles will tend to segregate and cluster.

## 4. Discussion of the results

The most important results of this analysis are that: (1) a suspension of rigid particles flowing in a plane Couette device with constant velocity at the wall is always stable, that is its concentration will tend to be uniform; (2) when the shear applied at the wall is kept constant, we predict that a disorder–order transition can occur, provided that $\sigma > 1$, where $\sigma$ is the non-dimensional parameter,

$$\sigma = \frac{\mu_0' \hat{E}_0}{\mu_0 \hat{D}_0}. \tag{48}$$

Here $\mu_0$ and $\mu_0'$ are the effective viscosities of the suspension, $\mu$, and its derivative, $\mathrm{d}\mu/\mathrm{d}\phi$, evaluated at the mean particle volume fraction $\phi_0$, while $\hat{D}_0$ and $\hat{E}_0$ are the transport coefficients appearing in the constitutive relation for the material flux of suspensions,

$$\mathbf{J} = -a^2 \hat{D}(\phi)\gamma\nabla\phi - a^2 \hat{E}(\phi)\nabla\gamma, \tag{49}$$

where $a$ is the particle radius, while $\gamma$ is the second invariant of the shear rate tensor. The first result agrees with the experimental findings of Tirumkudulu et al. (1999), who saw that when a suspension of monodisperse spheres is sheared within a Couette apparatus, particle segregation occurs when the Couette device is left partially empty, with a free surface on top. Otherwise, when the Couette apparatus is completely filled, the suspended particles tend to distribute uniformly without segregation.

The criterion for the onset of instability can be easily derived for the case of narrow Couette devices, where $\partial/\partial\hat{y} \gg \partial/\partial\hat{x}$, considering that the tangential stresses within a cross-section are constant, yielding:

$$F_0 = \frac{U_L}{L}\mu(1 + \hat{u}_y). \tag{50}$$

Therefore, given that after linearization, $\mu = \mu_0(1 + \hat{\mu}'_0 \widehat{\phi})$, we see that velocity and concentration fields are connected by $\widehat{u}_y = -\widehat{\mu}'_0 \widehat{\phi}$. Now, substituting this result into the continuity equation [cf. Eqs. (7) and (19)] and considering that $\gamma \simeq 1 + \widehat{u}_y$, we obtain

$$\widehat{\phi}_t + \hat{y}\widehat{\phi}_x = \lambda^2 \left( \widehat{D}_0 - \widehat{\mu}'_0 \widehat{E}_0 \right) \widehat{\phi}_{yy}. \tag{51}$$

From here we see that when $\sigma > 1$ the RHS becomes antidiffusive, determining the exponential growth of any infinitesimal perturbation.

Now, when the material flux is directed, as in our case, along a direction lying in the plane of shear, Leighton and Acrivos (1987b) proposed the following constitutive equation:

$$\mathbf{J} = -a^2 \widehat{D}_{\mathrm{LA}}(\phi)\gamma\nabla\phi - a^2 \widehat{E}_{\mathrm{LA}}(\phi)\nabla(\mu\gamma), \tag{52}$$

where $\mu$ is the effective suspension viscosity. In this case we find

$$\frac{1}{\sigma} = 1 + \frac{\widehat{D}_{\mathrm{LA}}}{\mu'_0 \widehat{E}_{\mathrm{LA}}} \tag{53}$$

showing that $\sigma < 1$ always. Analogously, the constitutive equation proposed by Phillips et al. (1992) reduces to Eq. (49) with $\widehat{D}(\phi) = \phi^2 K_\mu \mu'/\mu + \phi K_c$ and $\widehat{E}(\phi) = \phi^2 K_c$, where the coefficients $K_c$ and $K_\mu$ were considered as constant in Phillips et al. (1992), while they have been found to be concentration-dependent by Tetlow et al. (1998). Finally we obtain

$$\frac{1}{\sigma} = \frac{K_\mu}{K_c} + \frac{\mu_0}{\phi\mu'_0} \tag{54}$$

and, since $K_\mu > K_c$, here, too, we find that the system is always stable. This result, though, must be taken with a lot of suspicion, as the constitutive equations of suspension mechanics (with all the coefficients appearing in them) are still the subject of active investigations.

## Acknowledgements

During this work, R. Mauri and D.T. Papageorgiou were supported in part by the National Science Foundation, grants No. CTS-9711442 and DMS-990493, respectively.

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